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ON PRODUCTS OF TWO NILPOTENT SUBGROUPS OF A FINITE GROUP

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ABSTRACT

Let G be a finite group with an abelian Sylow 2-subgroup. Let A be a nilpotent subgroup of G of maximal order satisfying class $(A) \leqq k$, where k is a fixed integer larger than 1. Suppose that A normalizes a nilpotent subgroup B of G of odd order. Then AB is nilpotent. Consequently, if $F(G)$ is of odd order and A is a nilpotent subgroup of G of maximal order, then $F(G) \subseteq A$.

A. Introduction and notation

All groups in this paper are finite. We shall use the following notation.

$\mathcal{N}$	—	The set of non negative integers.
$\mathcal{P}$	—	The set of all prime numbers.
G	—	A finite group.
$F(G)$	—	The Fitting subgroup of G.
$\Phi(G)$	—	The Frattini subgroup of G.
$\pi(G)$	—	The set of primes dividing $\lvert G \rvert$.
$S_p(G)$	—	A p-Sylow subgroup of G.
$\mathcal{F}$	—	The set of all functions f s.t. $f\colon \mathcal{P} \to \mathcal{N} \cup \{\infty\}$.
class(G)	—	The nilpotency class of a nilpotent group G.

Let G be a finite group and let $f \in \mathcal{F}$. Define:

$\mathcal{A}(f, G) = \{A \mid A$ is of maximal order among all subgroups of G satisfying (a) A is nilpotent, and (b) for all $p \in \mathcal{P}$ class$(S_p(A)) \leqq f(p)\}$

$d(f, G) = \lvert A \rvert$ where $A \in \mathcal{A}(f, G)$.

REMARKS.

(a) $S_p(A)$ is nilpotent of class $\leqq 0 \Leftrightarrow S_p(A) = 1$.

(b) Clearly in considering $\mathcal{A}(f, G)$, we are interested in the restriction of f to $\pi(G)$, since if $p \not\in \pi(G)$ then class$(S_p(A)) = 0$.

(c) The statement: "class$(S_p(A)) \leqq \infty$" does not restrict $S_p(A)$. Therefore

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$\mathscr{A}(\infty, G)$ denotes the set of all nilpotent subgroups of G of maximal order.

(d) If f is a constant function $f \equiv a$, when $a \in \mathscr{N} \cup \{\infty\}$, $\mathscr{A}(f, G)$ will be denoted by $\mathscr{A}(a, G)$ and by definition: $\mathscr{A}(a, G) = \{A \mid A$ is of maximal order among all subgroups of G satisfying (a) A is nilpotent, and (b) $\text{class}(A) \leqq a\}$.

Using the notation above the following was proved in [1]: If G is a group of odd order, $A \in \mathscr{A}(1, G)$ and A normalizes a nilpotent subgroup B of G, then AB is nilpotent. If G is of even order, then the last result does not hold, unless extra conditions are imposed. The even case is also discussed in [1]. In [3] it was proved that if G is a finite group, $A \in \mathscr{A}(2, G)$ and A normalizes a nilpotent subgroup B of G, then AB is nilpotent.

It is natural to ask whether it is possible to generalize the above results for $k > 2$. For groups of odd order the positive answer is given in:

COROLLARY C.5. *Let G be a group of odd order, $A \in \mathscr{A}(k, G)$, $k \in \mathscr{N} \cup \{\infty\}$, $k \geqq 2$ and assume that A normalizes a nilpotent subgroup B of G. Then AB is nilpotent. Consequently, if $A \in \mathscr{A}(\infty, G)$, then $F(G) \subseteq A$.*

Corollary C.5 follows from

COROLLARY C.4. *Let G be a group with an abelian Sylow 2-subgroup. Let $A \in \mathscr{A}(k, G)$, $k \in \mathscr{N} \cup \{\infty\}$, $k \geqq 2$ and assume that A normalizes a nilpotent subgroup B of G of odd order. Then AB is nilpotent.*

Corollary C.4 is, in turn, an immediate result of the following theorem:

THEOREM C.3. *Let G be a group with an abelian Sylow 2-subgroup. Let B be a nilpotent subgroup of G of odd order, $f \in F$ and assume that either (1) or (2) holds:*

(1) *$f(p) \geqq 2$ for all $p \in \pi(B)$*

(2) *$f(p) \geqq 1$ for all $p \in \pi(B)$ and B is abelian.*

Then if $A \in \mathscr{A}(f, G)$ and A normalizes B, then AB is nilpotent.

Theorem C.3 yields also the following corollaries.

COROLLARY C.6. *Let G be a group of odd order and assume that $f \in \mathscr{F}$ satisfies $f(p) \geqq 2$ for all $p \in \pi(G)$. Then $d(f, G)$ and $|F(G)|$ have the same prime divisors.*

The next corollary is a generalization of a theorem of Burnside [2] for groups of odd order.

COROLLARY C.7. *Let $G = HK$ be a group of odd order, where H and K are π-Hall and π'-Hall subgroups of G, respectively. Then $d(\infty, H) > d(2, K)$ implies $O_\pi(G) \neq 1$.*

Corollary C.8 is a generalization for groups of odd order of a well known theorem stating that $F(G) = O_{p_1}(G) \times O_{p_2}(G) \times \cdots \times O_{p_k}(G)$, when $\pi(G) = \{p_1, p_2, \cdots, p_k\}$.

COROLLARY C.8. *Let G be a group of odd order and let $\pi(G) = \{\pi_1, \pi_2, \cdots, \pi_k\}$ be any partition of $\pi(G)$. If H_i denotes a π_i-Hall subgroup of G and $A_i \in \mathcal{A}(\infty, H_i)$ for $i = 1, \cdots, k$, then*

$$F(G) = \bigcap_{x \in G} A_1^x \times \bigcap_{x \in G} A_2^x \times \cdots \times \bigcap_{x \in G} A_k^x.$$

The proof of theorem C.3 depends on an important property of the group $GL(n, q)$, $q \in \mathcal{P}$, (Theorem B.7) which is obtained using methods and results of [5]. First a definition:

Let q be a fixed prime and suppose that $f \in \mathcal{F}$ satisfies $f(q) = 0$. We will say that q satisfies (property) α for f if for every $n \in \mathcal{N}$, $n > 0$, the following inequality holds: $d(f, GL(n, q)) < q^n$.

THEOREM B.7.

(a) *If q is an odd prime, then q satisfies α for $f \in \mathcal{F}$, s.t. $f(q) = 0$ and $f(2) \leqq 1$.*

(b) *If q is an odd prime, neither a Fermat-prime nor a Mersenne-prime, then q satisfies α for $f \in \mathcal{F}$ s.t. $f(q) = 0$ and $f(p) \leqq 1$ for all primes $\neq 2$.*

(c) *If $q = 2$, then q satisfies α for $f \in \mathcal{F}$ s.t. $f(2) = 0$ and $f(p) \leqq 1$ for all primes $p \neq r$, where r is a non-Mersenne-prime.*

A similar result was obtained in [3], where it was shown that any prime q satisfies α for $f \in \mathcal{F}$ s.t. $f(q) = 0$ and $f(p) = 2$ for all $p \in \mathcal{P}$, $p \neq q$.

B. On the property α

LEMMA B.1. *Let $G = HN$ be a group, where H and N are π-Hall and π'-Hall subgroups of G, respectively, and $N \lhd G$. Suppose $O_\pi(G) = 1$ and A is a group of H. Then for all $x \in N$, $A \cap H^x = C_A(x)$.*

PROOF. Let $x \in N$; then $C_A(x) \subseteq A \cap (C_A(x))^x \subseteq A \cap H^x$. Let $h \in A \cap H^x$, then $h = x^{-1}h_1x$ where $h_1 \in H$. Equivalently $h = h_1[h_1, x]$, but $[h_1, x] \in N$, so $h = h_1$ and $h = x^{-1}hx$. It follows that $h \in C_A(x)$.

LEMMA B.2. *Let $G = HN$ be a group, where H and N are q'-Hall and q-Sylow subgroups of G, respectively, and $N \lhd G$. Suppose that $O_{q'}(G) = 1$ and N is a minimal normal subgroup of G. Let $x \in N$; then:*

(a) $C_{Z(H)}(x) = 1$ *or equivalently, by* B.1, $Z(H) \cap H^x = 1$.

(b) $Z(H)$ *is cyclic.*

PROOF.

(a) Let $g \in C_{Z(H)}(x)$; then $g \in C_{Z(H)}(x^h)$ for every $h \in H$, so g centralizes $\langle x^H \rangle$, which is a normal subgroup of G included in N. By the minimality of N, $\langle x^H \rangle = N$. Now applying Lemma 1.2.3 of Hall-Higman we get that $g = 1$.

(b) Apply to the group $Z(H)N$ the theorem about the structure of a Frobenius complement (theorem 12.6.15 of [8]).

THEOREM B.3. *Let $G = HN$ be a group of odd order, where N is a minimal normal subgroup of G of order q^n, $q \in \mathcal{P}$, and H is a nilpotent q'-Hall subgroup of G. Let $O_{q'}(G) = 1$; then there exist $n_1, n_2 \in N$ which are not H-conjugate and such that $H \cap n_1^{-1}Hn_1 = H \cap n_2^{-1}Hn_2 = 1$.*

PROOF.

(a) Assume first that H is abelian. Applying B.2. (a) it follows that for all $n \in N$, $H \cap H^n = 1$. It is left to show that there are two non H-conjugate elements in N. But otherwise we would have $|H| = q^n - 1$, and this is impossible since G is of odd order.

(b) So we may assume that H is not abelian and hence N is not cyclic. We shall show that there exists a maximal subgroup $\hat{H}$ in H s.t. N is not minimal normal in $\hat{H}N$ and $Z(\hat{H})$ is non-cyclic.

In view of B.2. (b) it is sufficient to show that there exists a maximal subgroup $\hat{H}$ in H s.t. $Z(\hat{H})$ is non-cyclic. Since H is not abelian, there exists a p-Sylow subgroup P of H which is not cyclic. By theorem 9.5 of [7] there exists a non-cyclic normal subgroup L of P of order p^2. Define $\hat{H} = C_H(L)$. Using the N/C theorem and in view of the fact that by B.2. (b) $Z(H)$ is cyclic, it follows that $\hat{H}$ is maximal in H.

(c) Let N_1 be a minimal normal subgroup in $\hat{H}N$ s.t. $N_1 \not\subseteq N$ and let $d \in H \backslash \hat{H}$. It will be shown that:

$$N = N_1 \times d^{-1}N_1 d \times \cdots d^{-(p-1)}N_1 d^{p-1}.$$

By the minimality of N it follows that:

$$N = N_1 \cdot d^{-1}N_1 d \cdots d^{-(p-1)}N_1 d^{p-1}. \tag{1}$$

Let $\tilde{H}$ be the centralizer of N_1 in $\hat{H}$. Since $C_G(N) \subseteq N$, $\tilde{H} \cap d^{-1}\tilde{H}d \cap \cdots \cap d^{-(p-1)}\tilde{H}d^{p-1} = 1$. Now it is clear that if $1 \leqq i \leqq p-1$, then either $N_1 \cdot d^{-1}N_1d \cdots d^{-(i-1)}N_1d^{i-1} \cap d^{-i}N_1d^i$ is equal 1 or $d^{-i}N_1d^i \subseteq N_1 \cdot d^{-1}N_1d \cdots d^{-(i-1)}N_1d^{i-1}$. If for all i, $1 \leqq i \leqq p-1$, $N_1 \cdot d^{-1}N_1d \cdots d^{-(i-1)}N_1d^{i-1} \cap d^{-i}N_1d^i = 1$, we are through. So assume that i is the smallest integer s.t.: $N_1 \times d^{-1}N_1d \times \cdots \times d^{-(i-1)}N_1d^{i-1} \supseteq d^{-i}N_1d^i$. Let $d^{-i}n_id^i \in d^{-i}N_1d^i$, then $d^{-i}n_id^i = n_0d^{-1}n_1d \cdots d^{-(i-1)}n_{i-1}d^{i-1}$ and this representation is unique. If $h \in H$, it is easy to see that

(2) if $h \in \hat{H}$ ($h \in H \backslash \hat{H}$) then h normalizes (permutes) the factors in (1). Hence if $h \in C_{\hat{H}}(d^{-i}n_id^i)$, then h centralizes all the $d^{-k}n_kd^k$ in the representation of $d^{-i}n_id^i$. We can choose such a non identity element $d^{-k}n_kd^k$. Since $d^{-i}\tilde{H}d^i$ is normal in $d^{-i}\hat{H}d^i = \hat{H}$, it centralizes $\langle (d^{-k}n_kd^k)^{\hat{H}} \rangle$, which is a non identity subgroup of N normalized by $\hat{H}$. So $\langle (d^{-k}n_kd^k)^{\hat{H}} \rangle = d^{-k}N_1d^k$ yielding $d^{-i}\tilde{H}d^i = d^{-k}\tilde{H}d^k$, hence $\tilde{H} \lhd H$ and $\tilde{H} = 1$. But if $\tilde{H} = 1$, $O_{q'}(\hat{H}N_1) = 1$ and by B.2. (b) $Z(\hat{H})$ is cyclic in contradiction to part (b).

(d) Proof of the theorem by induction on $|G|$.

Consider $\tilde{H}N_1/\tilde{H}$, which is a minimal normal subgroup of $\hat{H}N_1/\tilde{H}$. By induction hypothesis there exist elements $n_1^{(1)}, n_2^{(1)} \in N_1$ s.t. $\hat{H} \cap n_1^{(1)-1}\hat{H}n_1^{(1)} = \hat{H} \cap n_2^{(1)-1}\hat{H}n_2^{(1)} = \tilde{H}$, where $n_1^{(1)}$ and $n_2^{(1)}$ are not $\hat{H}$-conjugate.

Defining:

$$n_1 = n_1^{(1)} \cdot d^{-1}n_2^{(1)}d \cdot \cdots d^{-(p-1)}n_2^{(1)}d^{p-1}$$

$$n_2 = n_2^{(1)} \cdot d^{-1}n_1^{(1)}d \cdot \cdots d^{-(p-1)}n_1^{(1)}d^{p-1}$$

we shall get that $H \cap n_1^{-1}Hn_1 = H \cap n_2^{-1}Hn_2 = 1$. It will be proved only that $H \cap n_1^{-1}Hn_1 = 1$, since the other equality is obtained similarly. By B.1 it is sufficient to prove that $C_H(n_1) = 1$. Let $ed^x \in C_H(n_1)$, where $e \in \hat{H}$ and $0 \leqq x \leqq p-1$. Then:

$$n_1 = (ed^x)^{-1}n_1ed^x = (ed^x)^{-1}n_1^{(1)}ed^x \cdot (ed^x)^{-1}d^{-1}n_2^{(1)}ded^x \ldots$$
$$\ldots (ed^x)^{-1}d^{-(p-1)}n_2^{(1)}d^{p-1}ed^x.$$

Assuming that $x > 0$, it follows by (2) that the N_1-component in the representation of n_1 is of one of the forms: $(ed^x)^{-1}d^{-\lambda}n_2^{(1)}d^{\lambda}ed^x$ or $(ed^x)^{-1}n_1^{(1)}ed^x$. But the last form is impossible, since then: $(ed^x)^{-1}n_1^{(1)}ed^x \in d^{-x}N_1d^x \cap N_1 = 1$. As $d^{\lambda}ed^x$ normalizes N_1, $d^{\lambda}ed^x \in \hat{H}$. Thus $n_1^{(1)}$ and $n_2^{(1)}$ are $\hat{H}$-conjugate, a contradiction. Therefore $x = 0$ and e centralizes n, hence also $n_1^{(1)}$. Thus $e \in \hat{H} \cap n_1^{(1)-1}\hat{H}n_1^{(1)} = \tilde{H}$. By the same argument since e centralizes $d^{-i}n_2^{(1)}d^i$, $e \in d^{-i}\tilde{H}d^i$ for every i, $1 \leqq i \leqq p-1$. So $e \in \tilde{H} \cap d^{-1}\tilde{H}d \cap \cdots \cap d^{-(p-1)}\tilde{H}d^{p-1}$, hence $e = 1$. To complete the proof it is left to show that n_1 and n_2 are not H-conjugate. Assume $n_1 = (ed^x)^{-1}n_2ed^x$, where $e \in \hat{H}$, $0 \leqq x \leqq p-1$. Assuming $x > 0$, by substitution we get: $n_1 = (ed^x)^{-1} n_2^{(1)} ed^x \cdot (ed^x)^{-1} d^{-1} n_1^{(1)} ded^x \cdots ed^x d^{-(p-1)} n_1^{(1)} d^{p-1} ed^x$. The N_1-component of n_1 in the last representation is of form: $(ed^x)^{-1} d^{-\mu} n_1^{(1)} d^{\mu} ed^x$, $1 \leqq \mu \leqq p-1$, since if it was of the form: $(ed^x)^{-1} n_2^{(1)} ed^x$, ed^x would normalize N_1, in contradiction to $x > 0$. Since $p > 2$, there are λ, μ $1 \leqq \lambda$, $\mu \leqq p-1$ s.t.: $d^{-\mu} n_2^{(1)} d^{\mu} = (ed^x)^{-1} d^{-\lambda} n_1^{(1)} d^{\lambda} ed^x$ or $n_2^{(1)} = (ed^{x-\mu})^{-1} d^{-\lambda} n_1^{(1)} d^{\lambda} ed^{x-\mu}$. It follows that $d^{\lambda} ed^{x-\mu} \in \hat{H}$, since by (2) $d^{\lambda} ed^{x-\mu}$ normalizes N_1. Thus $n_1^{(1)}$ and $n_2^{(1)}$ are $\hat{H}$-conjugate, a contradiction. Therefore $x = 0$. By equating the N_1-component in the two forms for n_1, we get $n_1^{(1)} = e^{-1} n_2^{(1)} e$ where $e \in \hat{H}$. But this is impossible, since $n_1^{(1)}$ and $n_2^{(1)}$ are not $\hat{H}$-conjugate.

THEOREM B.4. *Let G be a group of odd order and suppose that H is a nilpotent π-Hall subgroup of G. Then there exists $x \in G$ s.t. $H \cap H^x = O_\pi(G)$.*

PROOF. By induction on $|G|$.

(a) We can assume that $O_\pi(G) = 1$.

(b) We can assume that $G = HR$, where R is a normal π'-subgroup of G.

G is of odd order, hence by Feit-Thompson G is solvable. Consider the group $H \cdot O_{\pi'}(G)$; by Lemma 1.2.3 of Hall-Higman $O_\pi(HO_{\pi'}(G)) = 1$, so, if $HO_{\pi'}(G) \neq G$ the theorem follows by induction.

(c) We can assume that R is a minimal normal subgroup of G.

Let N be a minimal normal subgroup of G contained properly in R. Consider the group G/N; $O_\pi(G/N) = H^*N/N$ when H^* is a π-subgroup of G. By induction hypothesis there exist π-Hall subgroups of G, H_1 and H_2, s.t. $H_1N \cap H_2N = H^*N$. Clearly we can assume that $H_1 \cap H_2 = H^*$. Since $O_\pi(H^*N) = 1$, by induction hypothesis there exists $n \in N$ s.t. $H^* \cap n^{-1}H^*n = 1$. We shall see that $H_1 \cap n^{-1}H_2n = 1$. Indeed,

(3) $H_1 \cap n^{-1}H_2n \subseteq H_1N \cap n^{-1}H_2nN = H_1N \cap H_2N = H^*N$ and (3) implies (4)

(4) $H_1 \cap n^{-1}H_2n = (H_1 \cap H^*N) \cap (n^{-1}H_2n \cap H^*N) = H^* \cap n^{-1}H^*n = 1$, yielding (c).

(d) We have now the conditions of Theorem B.3 and Theorem B.4 follows.

The next theorem is cited from [6]. The proof of it is in [5] and [6].

THEOREM B.5. *Let G be a solvable group and let P be a p-Sylow subgroup of G. Suppose that either condition* (a) *or condition* (b) *holds*:

(a) *p is an odd non-Mersenne prime.*

(b) *$p = 2$ and $|G|$ is not divisible by a Fermat or Mersenne prime. Then there exists an $x \in G$* s.t. $P \cap P^x = O_p(G)$.

THEOREM B.6. *Let G be a solvable group with a nilpotent π-Hall subgroup H. Suppose that one of the following conditions is satisfied*:

(a) *$S_2(H)$ is abelian and non trivial.*

(b) *$S_{p'}(H)$ is abelian for a non-Mersenne odd prime p.*

(c) *$S_{2'}(H)$ is abelian and if $q \in \pi(G)\backslash\pi(H)$, then q is neither a Fermat prime nor a Mersenne-prime.*

Then there exists an $x \in G$ s.t. $H \cap H^x = O_\pi(G)$.

PROOF. By induction on $|G|$. Following parts (a), (b), (c) of Theorem B.4 we can assume that $O_\pi(G) = 1$ and $G = HR$, where R is a minimal normal subgroup of G. We can write $H = H_1 \times H_2$ where $(|H_1|, |H_2|) = 1$ and in case (a) H_1 is $S_2(H)$, in case (b) H_1 is $S_{p'}(H)$, and in case (c) H_1 is $S_{2'}(H)$. In all cases H_1 is abelian. Applying Theorems B.4 and B.5 to the group H_2R (B.4 in case (a) and B.5 in cases (b) and (c)) we get that there exists an $x \in R$ s.t. $H_2 \cap H_2^x = 1$, so clearly $H_2 \cap H^x = 1$. As $H_1 \subseteq Z(H)$, applying Lemma B.2. (a) to $G = HR$ we get $H_1 \cap H^x = 1$. Combining the two last results we get that $H \cap H^x = 1$.

THEOREM B.7.

(a) *If q is an odd prime, then q satisfies α for $f \in \mathcal{F}$,* s.t.: *$f(q) = 0$ and $f(2) \leqq 1$.*

(b) *If q is an odd prime, neither a Fermat-prime nor a Mersenne-prime, then q satisfies α for $f \in \mathcal{F}$* s.t. *$f(q) = 0$ and $f(p) \leqq 1$ for all primes $p \neq 2$.*

(c) *If $q = 2$, then q satisfies α for $f \in \mathcal{F}$* s.t. *$f(2) = 0$ and $f(p) \leqq 1$ for all primes $p \neq r$ where r is a non-Mersenne-prime.*

PROOF. Denote by V the elementary abelian group of order q^n and let $A \in \mathcal{A}(f, GL(n,q))$. Let G be the extension of V by A, $G = A \cdot V$. It follows then by Theorems B.4 and B.6 that there exists an $x \in G$ s.t. $A \cap A^x = 1$, whence $|A| < |V| = q^n$.

EXAMPLES.

(a) Let $f \in \mathcal{F}$ be defined by $f(p) = 10$ for $p \neq 5$ and $f(5) = 0$. Then 5 does not satisfy α for f, since $2^{11} \mid |GL(4,5)|$ and $2^{11} > 5^4$.

(b) Let $f \in \mathcal{F}$ be defined by $f(p) = 3$ for $p \neq 2$ and $f(2) = 0$. Then 2 does not satisfy α for f, since $3^4 \mid |(GL(6,2)|$ and $3^4 > 2^6$.

C. The main theorem and its corollaries

Let $q \in \mathcal{P}$, $a \in \mathcal{N} \cup \{\infty\}$ and $f \in \mathcal{F}$. Defining $f^{q,a} \in \mathcal{F}$ by

$$f^{q,a}(p)=\begin{cases} f(p) & \text{for} \quad p \neq q \\ a & \text{for} \quad p = q \end{cases}$$

we get:

THEOREM C.1. *Let* G *be a group,* $q \in \mathcal{P}$, $f \in \mathcal{F}$ s.t. $f(q)=0$ *and* $a \in \mathcal{N} \cup \{\infty\}$. *Consider the following statements*:

(a) q *satisfies* α *for* f.

(b.1) *If* $A \in \mathcal{A}(f^{q,a}, G)$ *and* A *normalizes a* q*-subgroup of* G, *then* AB *is nilpotent.*

(b.2) *If* $A \in \mathcal{A}(f^{q,a}, G)$ *and* A *normalizes an abelian* q*-subgroup of* G, *then* AB *is nilpotent.*

Then (a) *is equivalent to* (b.1) *for* $a \geqq 2$ *and* (a) *is equivalent to* (b.2) *for* $a = 1$.

PROOF.

Part A. (a) $\Rightarrow$ (b.1), (b.2).

The two cases will be proved simultaneously. Let $q \in \mathcal{P}, f \in \mathcal{F}$ s.t. $f(q)=0$ and let $a \in \mathcal{N} \cup \{\infty\}$. Suppose that q satisfies α for f and assume that either (b.1) or (b.2) does not hold. Let G be a counter example, i.e. there exists $A \in \mathcal{A}(f^{q,a}, G)$ normalizing a (abelian in case $a = 1$) q-subgroup B of G, and AB is not nilpotent. Choose G and B so that $|G|+|B|$ is minimal. Clearly we can assume that $G = AB$. The following notations will be used:

$$A_q = O_q(A), A_{q'} = O_{q'}(A) \text{ and } \Phi = \Phi(B).$$

I) $B = [B, A_{q'}]$ and in case (b.1) B is nilpotent of class at most 2.

By the minimality of $|B|$ it follows that $A_{q'}$ centralizes every proper subgroup of B which is normalized by A. In particular Φ is such a subgroup, so $A_{q'}$ operates on $V = B/\Phi$. It follows from theorem 5.2.3 [4, p. 177] that $V = C_V(A_{q'}) \times [V, A_{q'}]$. By the minimality of $|B|$ V cannot be A-decomposable. If $V = C_V(A_{q'})$ then AB is nilpotent, so $C_V(A_{q'}) = 1$ and $V = [V, A_{q'}]$ and it follows that $B = [B, A_{q'}]$. As B' is a proper A-invariant subgroup of B, it is centralized by $A_{q'}$. Using the three subgroups lemma we get from $[B', B, A_{q'}] = 1$ and $[A_{q'}, B', B] = 1$ that $[B, A_{q'}, B'] = 1$. But $B = [B, A_{q'}]$, so it follows that B is of class at most 2.

II) A_q centralizes B.

Let us consider the group $A_q V$ which is an extension of V by A_q. This is a q-group, so by a known property of nilpotent groups it follows that $[V, A_q] \neq V$. Since $[V, A_q]$ is A-invariant, it follows by the minimality of $|B|$

that $A_{q'}$ centralizes $[V, A_q]$. Since by part I $C_V(A_{q'}) = 1$, it follows that $[V, A_q]$ is trivial, hence $[B, A_q] \subseteq \Phi$. Applying the three subgroups lemma again, we get from $[B, A_q, A_{q'}] = 1$ and $[A_q, A_{q'}, B] = 1$ that $[A_{q'}, B, A_q] = 1$. But $[A_{q'}, B] = B$, so we get that A_q centralizes B.

III) Proof of part A.

Define $\bar{A} = A/C_A(B)$. If $|V| = |B/\Phi| = q^n$, then $\bar{A} \in GL(n, q)$. By II $\bar{A}$ is a q'-group and by (a) $\bar{A} \leqq d(f, GL(n, q)) < q^n = |V|$. Define $A^* = C_A(B)B$; clearly A^* is a nilpotent group. Since if H and K are commuting nilpotent groups then class $(HK) = \max\{\text{class}(H), \text{class}(K)\}$, it follows that in both cases (b.1) and (b.2) $\text{class}(S_p(A^*)) \leqq f^{q,\alpha}(p)$ for all $p \in \mathcal{P}$. Since $A \in \mathcal{A}(f^{q,\alpha}, G)$ it follows that $|A^*| \leqq |A|$. On the other hand it will be shown that $|A^*| > |A|$ and this leads to a contradiction. First, since $(B \cap C_A(B)\Phi)/\Phi$ is an A-invariant subgroup of V, it follows by arguments used in II that $B \cap C_A(B) \subseteq \Phi$. Now:

$$|A^*| = |C_A(B)B| = |B : B \cap C_A(B)|\,|C_A(B)| \geqq |B/\Phi|\,|C_A(B)| =$$

$$= |V|\,|C_A(B)| > |\bar{A}|\,|C_A(B)| = |A|.$$

Part B. (b.1), (b.2) $\Rightarrow$ (a).

Suppose (a) does not hold, then a group G will be constructed which is a counter-example to (b.1) and (b.2). Since (a) does not hold, q does not satisfy α for an f s.t. $f(q) = 0$, i.e. there exists an $n \in \mathcal{N}$, $n > 0$ s.t. $d(f, GL(n, q)) > q^n$. Let $A \in \mathcal{A}(f, GL(n, q))$; then A acts faithfully on an elementary abelian group V of order q^n. Define G as $A \cdot V$, which is the extension of V by A. Let $M \in \mathcal{A}(f^{q,\alpha}, G)$; then if (b.1) or (b.2) holds MV is nilpotent. Since $|M| \geqq |A| > q^n$, $MV \neq V$ and there is non trivial q'-element of G which centralizes V, in contradiction to the definition of G. The proof of Theorem C.1 is complete.

The results of Theorem B.7 can be substituted into C.1 to get criteria for a group to be nilpotent if it is a product of two of its nilpotent subgroups. Thus we get Theorem C.3 which requires also the following lemma.

LEMMA C.2. *Let A, B be nilpotent subgroups of G and suppose that A normalizes B. If for each $q \in \mathcal{P}$ s.t. $q \mid |B|$ $AO_q(B)$ is nilpotent then AB is nilpotent.*

PROOF. We may assume that $G = AB$. Assuming AB is not nilpotent, it follows that there exists a $p \in \mathcal{P}$ s.t. $O_p(A) \not\subseteq F(G)$. As $O_p(A)$ normalizes $O_p(B)$, consider the group $O_p(A)O_p(B)$. Since $O_p(A)O_p(B) \ntriangleleft G$, there exists a prime q, $q \neq p$, s.t. $O_q(B)$ does not normalize $O_p(A)O_p(B)$. It follows that $[O_p(A), O_q(B)] \neq 1$ in contradiction to the nilpotency of $AO_q(B)$.

THEOREM C.3. *Let G be a group with an abelian Sylow 2-subgroup. Let B be a nilpotent subgroup of G of odd order, $f \in \mathcal{F}$ and assume that either* 1) *or* 2) *holds:*

1) $f(p) \geqq 2$ *for all* $p \in \pi(B)$,
2) $f(p) \geqq 1$ *for all* $p \in \pi(B)$ *and B is abelian.*

Then if $A \in \mathcal{A}(f, G)$ and A normalizes B, then AB is nilpotent.

PROOF. Since $A \in \mathcal{A}(f, G)$ and $S_2(G)$ is abelian, we may assume that $f(2) \leqq 1$. As A normalizes the nilpotent group B, A normalizes $O_q(B)$ for all $q \in \pi(B)$. By Lemma C.2 it is sufficient to prove that for any $q \in \pi(B)$, $AO_q(B)$ is nilpotent. Let's fix a q, $q \in \pi(B)$; then $f(q) = a \geqq 1$ and we can write $f = \bar{f}^{q,a}$ for some function $\bar{f}$ defined by $\bar{f}(p) = f(p)$ for $p \neq q$ and $\bar{f}(q) = 0$. Since q is odd and $\bar{f}(2) \leqq 1$, by Theorem B.7. (a) q satisfies α for $\bar{f}$. Now substitute in C.1 q for q, $\bar{f}$ for f, A for A and $O_q(B)$ for B. We get, as either $a \geqq 2$ or $a = 1$ and $O_q(B)$ is abelian, that $AO_q(B)$, is nilpotent.

Theorem C.3 immediately yields

COROLLARY C.4. *Let G be a group with an abelian Sylow 2-subgroup. Let $A \in \mathcal{A}(k, G)$, $k \in \mathcal{N} \cup \{\infty\}$, $k \geqq 2$ and assume that A normalizes a nilpotent subgroup B of G of odd order. Then AB is nilpotent.*

By considering groups of odd order, we get

COROLLARY C.5. *Let G be a group of odd order, $A \in \mathcal{A}(k, G)$, $k \in \mathcal{N} \cup \{\infty\}$, $k \geqq 2$ and assume that A normalizes a nilpotent subgroup B of G. Then AB is nilpotent. Consequently, if $A \in \mathcal{A}(\infty, G)$, then $F(G) \subseteq A$.*

COROLLARY C.6. *Let G be a group of odd order, $f \in \mathcal{F}$, and assume that $f(p) \geqq 2$ for all $p \in \pi(G)$. Then $d(f, G)$ and $|F(G)|$ have the same prime divisors.*

PROOF. Let $A \in \mathcal{A}(f, G)$ and assume that $q \,\big|\, |F(G)|$ and $q \nmid |A|$. Let B be a minimal normal elementary abelian q-subgroup of G. By Theorem C.3 AB is a nilpotent subgroup of G of order larger than $|A|$. But $\operatorname{class}(S_p(AB)) \leqq f(p)$ for all p, in contradiction to the maximality of $|A|$. Thus every prime divisor of $|F(G)|$ divides $d(f, G)$.

On the other hand assume that $p \nmid |F(G)|$. Let $A \in \mathcal{A}(f, G)$; then by Theorem C.3 $AF(G)$ is nilpotent and hence $O_p(A) \subseteq C(F(G)) \subseteq F(G)$ yielding $p \nmid |A|$, as required.

COROLLARY C.7. *Let $G = HK$ be a group of odd order, where H and K are π-Hall and π'-Hall subgroups of G, respectively. Then $d(\infty, H) > d(2, K)$ implies $O_\pi(G) \neq 1$.*

PROOF. Define $f \in \mathcal{F}$ by: $f(p) = 2$ if $p \in \pi'$ and $f(p) = \infty$ if $p \in \pi$, and let $A \in \mathcal{A}(f, G)$. Suppose that $O_\pi(G) = 1$; then $F(G)$ is a π'-group and by Corollary C.6 A is a π'-group, contradicting $d(\infty, H) > d(2, K)$.

COROLLARY C.8. *Let G be a group of odd order and let $\pi(G) = \{\pi_1, \pi_2 \cdots \pi_k\}$ be any partition of $\pi(G)$. If H_i denotes a π_i-Hall subgroup of G and $A_i \in \mathcal{A}(\infty, H_i)$ for $i = 1, \cdots k$, then:*

$$F(G) = \bigcap_{x \in G} A_1^x \times \bigcap_{x \in G} A_2^x \times \cdots \bigcap_{x \in G} A_k^x$$

PROOF. Clearly $\bigcap_{x \in G} A_1^x \times \bigcap_{x \in G} A_2^x \times \cdots \bigcap_{x \in G} A_k^x \subseteq F(G)$. Let B_i denote $S_{\pi_i}(F(G))$. Since A_i normalizes B_i, by Theorem C.3 applied to H_i, A_iB_i is a nilpotent π_i-subgroup of H_i, hence $A_iB_i = A_i$ and so $B_i \subseteq A_i$, similarly $B_i \subseteq A_i^x$ for any $x \in G$. This proves that $B_i \subseteq \bigcap_{x \in G} A_i^x$ and so

$$F(G) \subseteq \bigcap_{x \in G} A_1^x \times \bigcap_{x \in G} A_2^x \times \cdots \bigcap_{x \in G} A_k^x.$$

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